

ON ISOCHRONOUS DERIVATIVES OF THE FIRST AND
SECOND ORDER IN SPACE DYNAMICS TASKS

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16. Abstract The problem is examined of calculating the first and second isochronous derivatives from the vector of state of dynamic system using its initial value. Use is made of the method of finding a fundamental solution of conjugate variational equations. This solution and the corresponding universal relationship for isochronous derivatives are found for the two-body problem in a form which is simple and suitable for computer programming. The simple form of these relationships was obtained for motion which differs from parabolic motion. Formulas are also given for isochronous derivatives using the gravitational parameter in the two-body problem.			
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ON ISOCHRONOUS DERIVATIVES OF THE FIRST AND SECOND ORDER IN SPACE DYNAMICS TASKS

B. Ts. Bakhshiyev and A. A. Sukhanov

INTRODUCTION

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In many problems of space dynamics, whose solution requires a linear approximation, it is necessary to calculate the isochronous derivatives of the first order from the phase coordinates using their values at the initial moment of time, and also using the values of certain constants in the right sides of the equations of motion (such as the gravitational parameter, equatorial radius, etc.). These derivatives are required when determining the trajectory of a spacecraft, for determining the reliability of predicting and the magnitude of the correcting pulse, for analyzing the perturbed motion and determining the error of numerical methods of celestial mechanics [1-9].

The second isochronous derivatives may be necessary for a more effective solution of these problems, and also when considering the influence of nonlinearity. For example, they are necessary for studying the influence of nonlinearity upon the reliability of the least square method [10] and when analyzing the perturbed motion in the second approximation [11].

The matrix of the first isochronous derivatives satisfies a system of variational equations and is related in a well-known way with the fundamental solution of the conjugate system [12], which in its turn plays an important role when optimizing space maneuvers and flight trajectories [13].

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In these problems, the presence of simple finite relationships are of great importance for isochronous derivatives and the solution of this conjugate system in the two-body problem. This is due to the fact that for slightly perturbed motion in several cases these relationships may be used to solve the problem with the necessary accuracy [2-8], and when it is necessary to consider the perturbations

* Numbers in margin refer to pagination of foreign text.

they make it possible to greatly increase the integration step of the variational system of equations (or the conjugate system) by using the method of Encke and Peano - Beker [14, 15]. It is desirable that these finite relationships have the following properties:

- a) convenience and simplicity for use on a computer, and great computational speed;
- b) universality, i.e., the lack of singularities for parabolic and circular orbits, and also in the case of rectilinear motion.

The formulas for isochronous derivatives of the first and second order, given in [1,2,4,8] and [11] do not satisfy condition b). In calculations using them for trajectories which are close to parabolic or rectilinear, large computational errors arise. More complex formulas for the derivatives of first order, obtained in [3, 16] by differentiation of the universal solution of the two-body problem are suitable for any Kepler motion, but they do not yield relationships for the second derivatives which can be readily used on computers or the explicit expressions for the fundamental solution of the conjugate system.

To calculate the isochronous derivatives, this article uses the relationship between the derivatives of the first order and the fundamental solution of the conjugate system (A. 1 and 2). In A.3 relationships are obtained for derivatives of the second order. In A.4 and 5 using the methods given in A. 1-3, formulas are obtained for calculating the first and second isochronous derivatives of the two-body problem which are suitable for any type of orbits. Thus, for orbits which differ from parabolic orbits, the relationships obtained may be written in a form which is apparently the simplest for use on a computer as compared with well-known methods. A.6 gives formulas for calculating the first and second derivatives using the gravitational parameter of the two-body problem.

A listing of all the formulas obtained is given in A.7.

We should also note that the integrals calculated in the Appendix are of independent value.

The authors plan to publish a preprint in the near future, which will ob-

tain the formulas given in this study in the form of a Fortran program.

1. MATRICES OF FIRST AND SECOND DERIVATIVES

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Let us examine a dynamic system described by the vector differential equation

$$\dot{\bar{x}} = \bar{f}(\bar{x}, t) \quad (1.1)$$

with the initial conditions $\bar{x}(t_0) = \bar{x}_0$. Here $\bar{x} = (x_1, \dots, x_n)^T$ is the n -dimensional vector-column^{*)}.

The matrix of the isochronous derivatives $\Phi = \Phi(t, t_0)$ of the phase coordinates may be written in the following form, using their initial values

$$\Phi = \frac{\partial \bar{x}}{\partial \bar{x}_0} = \left[\frac{\partial x_i}{\partial x_{0j}} \right]. \quad (1.2)$$

As is known [1], the matrix $\Phi(t, t_0)$ satisfies a variational system of equations

$$\dot{\Phi} = F \Phi, \quad \Phi(t_0, t_0) = I, \quad (1.3)$$

where $F = \partial \bar{f} / \partial \bar{x}$, I -- unit matrix.

The matrices of the second derivatives $\Psi^k = \Psi^k(t, t_0)$, $k = 1, \dots, n$ may be determined as follows:

$$\Psi^k = [\Psi_{ij}^k] = \left[\frac{\partial^2 x_k}{\partial x_{0i} \partial x_{0j}} \right]. \quad (1.4)$$

^{*)} Below all the vectors will be assumed to be vectors-columns; the sign " T " designates transposition.

The study [9] gives the differential equations which are satisfied by the matrix

$$\Psi^k.$$

2. METHOD OF OBTAINING THE FIRST DERIVATIVES IN SOLVING THE CONJUGATE SYSTEM

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Let us look for the matrix Φ as the solution of the system of equations

$$A\Phi = A_0, \quad (2.1)$$

where $A = A(t)$ is a certain nondegenerate matrix of the n -th order;

$A_0 = A(t_0)$. We shall use A_i, A_{0i} to designate the rows of the matrix A, A_0 . Differentiating (2.1) with respect to time, taking into account (1.3), we find that A_i satisfies the conjugate variational system of equations

$$\dot{A}_i = -A_i F, \quad A_i(t_0) = A_{0i}, \quad i = 1, \dots, n. \quad (2.2)$$

Thus, the determination of the matrix Φ is reduced to finding the fundamental solution $A(t)$ of the conjugate variational system of equations.

Let us assume that m independent first integrals of the system (1.1) are known:

$$V_i(\bar{x}) = V_i(\bar{x}_0), \quad i = 1, \dots, m. \quad (2.3)$$

Differentiating (2.3) with respect to \bar{x}_0 at $t = \text{const}$, we obtain

$$\frac{\partial V_i}{\partial \bar{x}_0} \Phi = \frac{\partial V_i}{\partial \bar{x}}, \quad i = 1, \dots, m. \quad (2.4)$$

It is thus apparent that the gradients of the first integrals of (1.1)

$$A_i = \frac{\partial V_i}{\partial \bar{x}}, \quad i = 1, \dots, m \quad (2.5)$$

are the m row of the matrix A . In order to obtain the remaining $n-m$ rows, it is necessary to find $n-m$ solutions of the system (2.2), which are independent

of each other and with the solutions (2.5). For the case when the system (1.1) is reduced to the Hamiltonian form and $n-1$ of its independent first integrals are known, the last row of the matrix A may be found using the following theorem [12] (exclusion method).

Charnyy method. Let us assume $A_i, i=1, \dots, n-1$ -- gradients (2.5) of the independent first integrals of the system (1.1) is reduced to the Hamiltonian form. Then the n -dimensional row

$$A_n = [\bar{y} - \bar{f} \int_{t_0}^t \lambda(t) dt]^T J \quad (2.6)$$

(Here $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is the matrix of n th order) is the solution of the system (2.2), if the n -dimensional vector \bar{y} and the function $\lambda(t)$ satisfy the following relationships:

$$A_i \bar{y} = C_i, \quad i=1, \dots, n-1, \quad (2.7)$$

$$\dot{\bar{y}} - F \bar{y} = \lambda(t) \bar{f}, \quad (2.8)$$

where C_i are the arbitrary constants. Thus, the matrix A , whose row is

$A_i, i=1, \dots, n$, is degenerate then and only then, when the following condition is satisfied

$$\sum_{i=1}^{n-1} C_i \lambda_i \neq 0, \quad (2.9)$$

where λ_i are the expansion coefficients of the Hamiltonian function gradient with respect to the rows $A_i, i=1, \dots, n-1$.

3. OBTAINING THE SECOND DERIVATIVES

Let us find the expressions for the matrices (1.4) when the fundamental solution of the system (2.2) is known in the form

$$A_\ell = A_\ell(\bar{x}, \bar{x}_0, t, t_0), \quad \ell=1, \dots, n.$$

Let us write (2.1) using the elements of the corresponding matrices:

$$\sum_{p=1}^n A_{lp} \Phi_{pi} = A_{oi} , \quad l, i = 1, \dots, n.$$

Differentiating this with respect to x_{oj} , we obtain

$$\sum_{p,j=1}^n (D_{pj}^l \Phi_{pj} + E_{pj}^l) \Phi_{pi} + \sum_{p=1}^n A_{lp} \Psi_{ij}^p = D_{oi}^l + E_{oi}^l, \quad (3.1)$$

where $D_{pj}^l, E_{pj}^l, D_{oi}^l, E_{oi}^l$ are elements of the matrix

$$D^l = \frac{\partial A^T}{\partial \bar{x}}, \quad E^l = \frac{\partial A^T}{\partial \bar{x}_o}, \quad D_o^l = (D^l)_{\bar{x}=\bar{x}_o, t=t_o}, \quad E_o^l = (E^l)_{\bar{x}=\bar{x}_o, t=t_o}. \quad (3.2)$$

Let us use a_{kl} to designate the elements of the matrix A^{-1} . Multiplying (3.1) by a_{kl} , and summing the relationship obtained with respect to l , with allowance for the fact that

$$\sum_{l,p=1}^n a_{kl} A_{lp} \Psi_{ij}^p = \Psi_{ij}^k$$

and changing to matrix form, we find the formulas for the second derivatives (1.4):

$$\Psi^k = \sum_{l=1}^n a_{kl} [D_o^l + E_o^l - \Phi^T (E^l + D^l \Phi)], \quad k=1, \dots, n. \quad (3.3)$$

We should note that if each of the matrices D^l, E^l may be substituted in (3.2) for certain l in the form of two components, so that the second components after substitution in (3.3) give an identical zero, then we shall use \bar{D}^l, \bar{E}^l in this case to designate the first components. This situation occurs for the two-body problem (see A.5).

4. FIRST DERIVATIVES FOR THE TWO-BODY PROBLEM

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In this section, formulas are obtained for the first derivatives which do

not have singularities for any types of Kepler orbits.

For the two-body problem, the vectors \bar{x}, \bar{p} and the matrix F have the form

$$\bar{x} = \begin{bmatrix} \bar{r} \\ \bar{v} \end{bmatrix}, \quad \bar{p} = \begin{bmatrix} \bar{v} \\ -\nu \bar{r} \end{bmatrix}, \quad F = \begin{bmatrix} 0 & I \\ G & 0 \end{bmatrix}, \quad (4.1)$$

where

$$\nu = \frac{\mu}{r^3}, \quad G = \nu \left(3 \frac{\bar{r} \bar{r}^T}{r^2} - I \right), \quad (4.2)$$

where \bar{r}, \bar{v} are the vectors of the coordinates and the velocities of the point, $r = |\bar{r}|$, $v = |\bar{v}|$, μ -- gravitational parameter of the attracting center. We first obtain the first five rows of matrix A , which are independent of each other for any Kepler motion. We use the first integrals of the two-body problem for this:

$$\bar{C} = \bar{r} \times \bar{v}, \quad (4.3)$$

$$\bar{\psi} = -\mu \frac{\bar{r}}{r} + \bar{v} \times \bar{C}, \quad (4.4)$$

$$h = v^2 - \frac{2\mu}{r}. \quad (4.5)$$

Let us assume \bar{p}_1, \bar{p}_2 are certain constant three-dimensional vectors. Multiplying each of the equations (4.3) and (4.4) in a scalar manner by \bar{p}_1, \bar{p}_2

and differentiating the relations obtained and (4.5) with respect to \bar{r}, \bar{v} , according to (2.5), we find the first five rows of the matrix A :

$$\left. \begin{aligned} A_i &= [\bar{w}_i^T, \bar{u}_i^T], \\ A_{i+2} &= [(\nu \bar{r} \times \bar{u}_i + \bar{v} \times \bar{w}_i)^T, (\bar{p}_i \times \bar{C} - \bar{r} \times \bar{w}_i)^T], \\ A_5 &= [\nu \bar{r}^T, \bar{v}^T] \end{aligned} \right\} i=1,2 \quad (4.6)$$

where

$$\bar{u}_i = \bar{p}_i \times \bar{r}, \quad \bar{w}_i = \bar{v} \times \bar{p}_i, \quad i=1,2. \quad (4.7)$$

For all types of orbits, except for rectilinear orbits, the rows $A_1 - A_5$ are independent for any non-colinear vectors \bar{p}_1, \bar{p}_2 . In the case of

rectilinear orbits and orbits close to them, for the linear independence of these rows, it is sufficient, for example, that the vectors $\bar{z}, \bar{p}_1, \bar{p}_2$ be linearly independent of each other (this may readily be established by substituting $\bar{v} = a\bar{z}$ in (4.6), where a is an arbitrary factor). Therefore, for the universality of formulas (4.6), it is advantageous to select \bar{p}_1, \bar{p}_2 from this condition in every case. As \bar{p}_1, \bar{p}_2 , we may select, for example, two unit vectors of the system of coordinates corresponding to the components of the vector \bar{z} which are the smallest in terms of modulus.

In order to find the row A_6 , we may use the Charnyy theorem. We shall search for a vector \bar{y} which satisfies (2.7) and (2.8) in the form

$$\bar{y} = \begin{bmatrix} \alpha_1 \bar{z} + \beta_1 \bar{v} \\ \alpha_2 \bar{z} + \beta_2 \bar{v} \end{bmatrix}. \quad (4.8)$$

Substituting (4.8) into (2.8), with allowance for (1.1), (4.1) and combining the coefficients at \bar{z}, \bar{v} , we obtain

$$\left. \begin{aligned} (\dot{\alpha}_1 - \beta_1 \nu - \alpha_2) \bar{z} + (\alpha_1 + \dot{\beta}_1 - \beta_2 - \lambda) \bar{v} &= 0, \\ (\dot{\alpha}_2 - \beta_2 \nu - 2\alpha_1 \nu - 3\beta_1 \nu \frac{\bar{z}^T \bar{v}}{\bar{z}^2} + \lambda \nu) \bar{z} + (\alpha_2 + \dot{\beta}_2 + \beta_1 \nu) \bar{v} &= 0. \end{aligned} \right\} \quad (4.9)$$

If the vectors \bar{z}, \bar{v} are non-colinear, then it follows from (4.9) that

$$\left. \begin{aligned} \dot{\alpha}_1 - \beta_1 \nu - \alpha_2 &= 0, \\ \alpha_1 + \dot{\beta}_1 - \beta_2 &= \lambda, \\ \dot{\alpha}_2 - \beta_2 \nu - 2\alpha_1 \nu - 3\beta_1 \nu \frac{\bar{z}^T \bar{v}}{\bar{z}^2} &= -\lambda \nu, \\ \alpha_2 + \dot{\beta}_2 + \beta_1 \nu &= 0. \end{aligned} \right\} \quad (4.10)$$

For rectilinear motion, we shall find the vector \bar{y} and the function λ from (4.8) and (4.10).

Solving system (4.10), we find

$$\alpha_1 = \bar{a}^T \bar{z} + 2\varepsilon_1, \quad (4.11)$$

$$\beta_2 = -\bar{a}^T \bar{c} - \varepsilon_1, \quad (4.12)$$

$$\alpha_2 + \beta_1 \nu = \bar{a}^T \bar{v}, \quad (4.13)$$

$$\lambda = 2\bar{a}^T \bar{c} + 3\varepsilon_1 + \beta_1, \quad (4.14)$$

where \bar{a} , ε_1 are arbitrary constants. We may readily determine that the coefficients α_2 and β_1 are included in (2.6) and (2.7) only in the form of the sum (4.13). This makes it possible to set in (4.8)

$$\beta_1 = 0. \quad (4.15)$$

Thus, the vector \bar{y} , found using the formulas (4.8), (4.11) - (4.15), satisfies the condition (2.8) of the Charny theorem. It may be readily seen that this vector will also satisfy the condition (2.7) if the constants $C_1 - C_5$ are related with the constants \bar{a} , ε_1 by the formulas

$$\left. \begin{aligned} C_i &= \varepsilon_1 \bar{c}^T \bar{p}_i, \\ C_{i+2} &= (\bar{a} \times \bar{c})^T (\bar{p}_i \times \bar{c}), \\ C_5 &= -\varepsilon_1 h - \bar{a}^T \bar{\psi}. \end{aligned} \right\} i=1,2 \quad (4.16)$$

Since the row A_5 is a gradient of the Hamiltonian function (i.e., in (2.9) $\lambda_i = 0$ at $i \neq 5, \lambda_5 = 2$), then (2.9) for the nondegenerate nature of the matrix A in the case considered may be reduced to the inequality $C_5 \neq 0$. We set

$$\bar{a} = \varepsilon_2 \bar{\psi}, \quad (4.17)$$

where ε_2 is an arbitrary constant. According to (4.16), for the nondegenerate nature of the matrix A, the constants $\varepsilon_1, \varepsilon_2$ must satisfy the condition /13

$$\varepsilon_1 h + \varepsilon_2 \psi^2 \neq 0 \quad (\psi = |\bar{\psi}|) \quad (4.18)$$

(this condition is satisfied for any Kepler motion, since h and $\bar{\psi}$ simultaneously vanish).

Substituting the values of \bar{y} , λ found from (4.8), (4.11) - (4.15), (4.17) into (2.6), we find the sixth row of the matrix A:

$$A_6 = \varepsilon_1 A_6^1 + \varepsilon_2 A_6^2, \quad (4.19)$$

where

$$A_6^1 = [-3\nu(t-t_0)\bar{z}^T + \bar{u}^T, 2\bar{z}^T - 3(t-t_0)\bar{u}^T], \quad (4.20)$$

$$A_6^2 = [(\mu\dot{z} - 2\nu s)\bar{z}^T + \dot{s}\bar{u}^T, \dot{s}\bar{z}^T - 2s\bar{u}^T]. \quad (4.21)$$

Here we use the notation

$$\dot{z} = \frac{\bar{z}^T \bar{u}}{\bar{z}}, \quad (4.22)$$

$$\dot{s} = \bar{u}^T \bar{z} = C^2 - \mu z \quad (C = |\bar{C}|), \quad (4.23)$$

$$s = \bar{u}^T \int_{t_0}^t \bar{z} dt = C^2(t-t_0) - \mu \int_{t_0}^t z dt. \quad (4.24)$$

It is shown in the Appendix that the integral in (4.19) is calculated in the form of a converging series

$$R = \int_{t_0}^t z dt = \frac{1}{\sqrt{\mu}} \left(z^2 \tau - z z' \tau^2 + \frac{z z'' + z'^2}{3} \tau^3 + \sigma \right), \quad (4.25)$$

$$\sigma = 2z^4 \sum_{n=2}^{\infty} \left[\frac{2^{2(n-1)}(z''^2 - d z'^2) - z''}{2n+1} \tau - (2^{2(n-1)} z'' - 1) z' \right] \frac{(-d \tau^2)^{n-2}}{(2n)!}, \quad (4.26)$$

where

$$d = -\frac{h}{\mu} = \frac{2}{z} - \frac{v^2}{\mu}, \quad (4.27)$$

$$\tau = d\sqrt{\mu}(t-t_0) + \frac{\bar{z}^T \bar{u} - \bar{z}_0^T \bar{u}_0}{\sqrt{\mu}} \quad (4.28)$$

and the control time,

$$z' = \frac{dz}{d\tau} = \frac{\bar{z}^T \bar{u}}{\sqrt{\mu}}, \quad (4.29)$$

$$z'' = \frac{d^2 z}{d\tau^2} = -d z + 1. \quad (4.30)$$

According to the Appendix, at $\alpha \neq 0$ (the motion differs from parabolic motion), the integral (4.25) may be calculated using the final formula

$$R = \frac{3\sqrt{\mu}(t-t_0) - p\tau - \gamma\gamma' + \gamma_0\gamma'_0}{2\alpha\sqrt{\mu}}, \quad (4.31)$$

where $p = C^2/\mu$ is the focal parameter.

Let us consider the problem of selecting the constants $\varepsilon_1, \varepsilon_2$ in (4.19). In the general case, for nondegenerate conditions on any orbits of the matrix A, consisting of the rows (4.6), (4.19), according to (4.18), we could take

$\varepsilon_1 = \text{sign}(h), \varepsilon_2 = 1$. However, for a more effective calculation of the row A_6 , it is advantageous to select the constants from the following conditions:

at $|h| > \varepsilon\mu/\gamma_0$ ($\varepsilon > 0$ -- the given small quantity) we assume

$\varepsilon_1 = 1, \varepsilon_2 = 0$, and at $|h| \leq \varepsilon\mu/\gamma_0$, i.e., if the motion is parabolic or close to it, we set $\varepsilon_1 = 0, \varepsilon_2 = 1$. In the first case of the row A_6 , we use the simple formula (4.20) to perform the calculation; in the second case -- we use formula (4.21). We should note that with this selection of the constants $\varepsilon_1, \varepsilon_2$ formula (4.31) cannot be used. The series is calculated only for values of α which are small in terms of the modulus, which leads to their rapid convergence.

5. SECOND DERIVATIVES FOR THE TWO-BODY PROBLEM

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In order to calculate the second isochronous derivatives for known matrices A and Φ , according to (3.3) it is sufficient to find the matrices D^l, E^l , $l = 1, \dots, 6$, determined from (3.2). We introduce the notation

$$\left. \begin{aligned} p_l &= \bar{\tau}^T \bar{p}_l, \quad \dot{p}_l = \bar{v}^T \bar{p}_l, \\ P_l &= \begin{bmatrix} 0 & p_{lz} & -p_{ly} \\ -p_{lz} & 0 & p_{lx} \\ p_{ly} & -p_{lx} & 0 \end{bmatrix}, \\ Q_l &= \dot{p}_l I + \bar{v} \bar{p}_l^T - 2\bar{p}_l \bar{v}^T, \end{aligned} \right\} l = 1, 2 \quad (5.1)$$

$$R_\ell = \bar{\tau} \bar{p}_\ell^T + \bar{p}_\ell \bar{\tau}^T.$$

Differentiating the rows (4.6) with respect to $\bar{\tau}, \bar{U}$, we obtain -- with allowance for the notation in (4.2) and (5.1) -- the first five matrices D^ℓ :

$$\left. \begin{aligned} D^\ell &= \begin{bmatrix} 0 & P_\ell \\ P_\ell^T & 0 \end{bmatrix}, \\ D^{\ell+2} &= \begin{bmatrix} -\nu R_\ell + \rho_\ell G & Q_\ell \\ Q_\ell^T & R_\ell - 2\rho_\ell I \end{bmatrix}, \\ D^5 &= \begin{bmatrix} -G & 0 \\ 0 & I \end{bmatrix}. \end{aligned} \right\} \ell=1,2 \quad (5.2)$$

Since the rows (4.6) do not depend on $\bar{\tau}_0, \bar{U}_0$, according to (3.2),

$$E^\ell \equiv 0, \quad \ell = 1, \dots, 5.$$

When finding the matrices D^6, E^6 , we shall assume that the selection of the constants $\varepsilon_1, \varepsilon_2$ in (4.19) is performed using the algorithm used in A.4.

Differentiating (4.20), we obtain for orbits different from parabolic orbits

$(|h| > \varepsilon \mu / \tau_0):$

$$\left. \begin{aligned} D_1^6 &= \begin{bmatrix} 3(t-t_0)G & I \\ 2I & -3(t-t_0)I \end{bmatrix}, \\ E_1^6 &\equiv 0. \end{aligned} \right\} \quad (5.3) \quad /16$$

Now let us find the matrices D^6, E^6 for parabolic orbits and orbits close to them $(|h| \leq \varepsilon \mu / \tau_0)$. We note that, if the inequality $C_5 \neq 0$ -- C_5 is determined from (4.16) at $\varepsilon_1 = 0$ -- is satisfied for the given values of the initial phase coordinates $\bar{\tau}_0, \bar{U}_0$, then for a fixed constant \bar{a} , it is satisfied also in a certain vicinity of these values (in view of the continuity of the function (4.16) with respect to $\bar{\tau}_0, \bar{U}_0$). Therefore, when finding the derivatives of (4.21), we shall assume that the constant \bar{a} does not depend on $\bar{\tau}_0, \bar{U}_0$, although it equals numerically (4.17). Let us write the row (4.21), using (4.17) - (4.24) in the form

$$A_6^2 = [-(\bar{\psi}^T \bar{U} + 2\nu \bar{\psi}^T \bar{R}) \bar{\tau}^T + (\bar{\psi}^T \bar{\tau}) \bar{U}^T, (\bar{\psi}^T \bar{\tau}) \bar{\tau}^T - 2(\bar{\psi}^T \bar{R}) \bar{U}^T], \quad (5.4)$$

where $\bar{R} = \int_{t_0}^t \bar{\gamma} dt$. We first find $\bar{\psi}^T \left(\frac{\partial \bar{R}}{\partial \bar{x}_0} \right)_c$ then (where "c" designates the complete derivative, i.e., taking into account the dependence

$\bar{x}(\bar{x}_0)$). As is shown in the Appendix, the integral \bar{R} may be represented in the form

$$2h\bar{R} = \bar{\varphi} - \bar{\varphi}_0, \quad (5.5)$$

where

$$\bar{\varphi} = \bar{\varphi}(\bar{\gamma}, \bar{u}, t) = 3(t-t_0)\bar{\psi} - (\bar{\gamma}^T \bar{u})\bar{\gamma} + 2\bar{\gamma}^2 \bar{u}, \quad (5.6)$$

From (5.5) $\bar{\varphi}_0 = \bar{\varphi}(\bar{\gamma}_0, \bar{u}_0, t_0)$.

$$2h\bar{\psi}^T \left(\frac{\partial \bar{R}}{\partial \bar{x}_0} \right)_c = \bar{\rho}^T \Phi - \bar{\rho}_0^T, \quad (5.7)$$

where

$$\begin{aligned} \bar{\rho}^T &= \bar{\rho}^T(\bar{\gamma}, \bar{u}, t) = \bar{\psi}^T \left(\frac{\partial \bar{\varphi}}{\partial \bar{x}} - 2\bar{\psi}^T \bar{R} \frac{\partial h}{\partial \bar{x}} \right), \\ \bar{\rho}_0 &= \bar{\rho}(\bar{\gamma}_0, \bar{u}_0, t_0) \end{aligned} \quad (5.8)$$

(This equation may be written as $\bar{R}(0) = \bar{0}$). Differentiating (4.5), (5.6) with respect to $\bar{\gamma}, \bar{u}$ and substituting in (5.8), we obtain

$$\bar{\rho} = A_6^T + 3h(t-t_0)\bar{\eta} + \bar{\xi}, \quad (5.9)$$

where A_6 is the row (4.19), in which $\varepsilon_1 = -C^2$, $\varepsilon_2 = 2$,

$$\bar{\eta} = \begin{bmatrix} \bar{u} \times \bar{C} \\ \bar{C} \times \bar{\gamma} \end{bmatrix}, \quad (5.10)$$

$$\bar{\xi} = \begin{bmatrix} -(5\frac{\mu}{\gamma} + u^2)(\bar{\gamma}^T \bar{u})\bar{\gamma} + (3\mu\gamma - 2\gamma^2 u^2 + 3(\bar{\gamma}^T \bar{u})^2)\bar{u} \\ (\mu\gamma + \gamma^2 u^2 + (\bar{\gamma}^T \bar{u})^2)\bar{\gamma} - 2\gamma^2(\bar{\gamma}^T \bar{u})\bar{u} \end{bmatrix}. \quad (5.11)$$

According to (2.1),

$$A_6 \Phi - A_{06} = 0. \quad (5.12)$$

It may be seen from (5.7) that at $h = 0$ the following equation must be satisfied

$$\bar{\xi}_p^T \Phi - \bar{\xi}_0^T = 0, \quad (5.13)$$

where $\bar{\xi}_p = (\bar{\xi})_{h=0}$, i.e., according to A.2, the row $\bar{\xi}_p^T$ in this case must satisfy the conjugate variational system of equations (2.2). Substituting $v^2 = 2\mu/\tau$ in (5.11) (according to 4.5), we obtain

$$\bar{\xi}_p^T = \begin{bmatrix} -7 \frac{\mu}{\tau} (\bar{\tau}^T \bar{v}) \bar{\tau} - (\mu \tau - 3(\bar{\tau}^T \bar{v})^2) \bar{v} \\ (3\mu \tau + (\bar{\tau}^T \bar{v})^2) \bar{\tau} - 2\tau^2 (\bar{\tau}^T \bar{v}) \bar{v} \end{bmatrix}. \quad (5.14)$$

Substituting (5.14) in equation (2.2), where the matrix F is determined from (4.1),

we see that $\bar{\xi}_p^T$ actually satisfies the conjugate variational system of equations and, consequently, equation (5.13) is satisfied. It may be readily seen that

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$$\bar{\xi} = \bar{\xi}_p + h \begin{bmatrix} -(\bar{\tau}^T \bar{v}) \bar{\tau} - 2\tau^2 \bar{v} \\ \tau^2 \bar{\tau} \end{bmatrix}. \quad (5.15)$$

We find from (5.7), (5.9), (5.10), (5.12), (5.13), (5.15)

$$2\bar{\psi}^T \left(\frac{\partial \bar{R}}{\partial \bar{x}_0} \right)_c = \bar{\xi}^T \Phi - \bar{\xi}_0^T, \quad (5.16)$$

where

$$\bar{\xi} = \bar{\xi}(\bar{\tau}, \bar{v}, t) = \begin{bmatrix} 3(t-t_0) \bar{v} \times \bar{C} - (\bar{\tau}^T \bar{v}) \bar{\tau} - 2\tau^2 \bar{v} \\ 3(t-t_0) \bar{C} \times \bar{\tau} + \tau^2 \bar{\tau} \end{bmatrix}, \quad (5.17)$$

$$\bar{\xi}_0 = \bar{\xi}(\bar{\tau}_0, \bar{v}_0, t_0).$$

Differentiating (5.4) with respect to $\bar{\tau}, \bar{v}, \bar{\tau}_0, \bar{v}_0$ and assuming that the vector $\bar{\psi}$ is constant, in agreement with the note made before the relationship (5.4), we obtain

$$D_2^6 = D_{21}^6 + D_{22}^6, \quad (5.18)$$

where, with allowance for the notation in (4.2), (4.22) = (4.24), (5.17),

$$D_{21}^6 = \begin{bmatrix} 2sG + \mu \dot{\tau} I + \bar{v} \bar{\psi}^T & \dot{s} I - \bar{\tau} \bar{\psi}^T \\ \dot{s} I - \bar{\tau} \bar{\psi}^T & -2sI \end{bmatrix}, \quad (5.19)$$

$$D_{22}^6 = - \begin{bmatrix} \dot{v} \bar{\tau} \\ \bar{v} \end{bmatrix} \bar{\xi}^T \quad (5.20)$$

$$E_2^6 = \begin{bmatrix} \dot{v} \bar{\tau} \\ \bar{v} \end{bmatrix} \bar{\xi}_0^T. \quad (5.21)$$

We should note that the matrices D_2^6 , E_2^6 equal the derivative (3.2) within an accuracy of terms which, according to (5.12), (5.13), may be disregarded in (3.3) (see A.3).

6. DERIVATIVES WITH RESPECT TO THE GRAVITATIONAL PARAMETER IN THE TWO-BODY PROBLEM

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In the two-body problem for isochronous derivatives of the first order of the vector of the phase coordinates $\bar{x} = [\bar{\tau}^T, \bar{v}^T]^T$ with respect to the gravitational parameter μ , the following formulas hold

$$\frac{\partial \bar{x}}{\partial \mu} = \frac{1}{3\mu} (\bar{x} - \Phi \bar{x}_0), \quad (6.1)$$

where Φ is the matrix of the derivatives (1.2).

Relationship (6.1) is a result of the integrals obtained in [7] for variational systems of equations. The study [5] found these relationships by integration of the equations for the desired derivatives.

Differentiating (6.1) with respect to \bar{x}_0 and μ , we obtain the formulas for the derivatives of second order

$$\frac{\partial^2 \bar{x}}{\partial \mu \partial \bar{x}_0} = \frac{\partial \Phi}{\partial \mu} = - \frac{1}{3\mu} \begin{bmatrix} \bar{x}_0^T \Psi^1 \\ \vdots \\ \bar{x}_0^T \Psi^6 \end{bmatrix}, \quad (6.2)$$

$$\frac{\partial^2 \bar{x}}{\partial \mu^2} = -\frac{1}{3\mu} \left(2 \frac{\partial \bar{x}}{\partial \mu} + \frac{\partial \Phi}{\partial \mu} \bar{x}_0 \right), \quad (6.3)$$

where Ψ^K ($K=1, \dots, 6$) are the matrices of the derivatives (.14)

7. SUMMARY OF FORMULAS

Let us calculate the formulas which make it possible to solve the conjugate variational system of equations and to calculate the isochronous derivatives of the first and second order for the two-body problem.

The matrix of the isochronous derivatives of the first order (1.2) may be found from the system of linear algebraic equations (2.1), where /20

$$A = A(\bar{r}, \bar{v}, t), \quad A_0 = A(\bar{r}_0, \bar{v}_0, t_0)$$

are the matrices of 6th order. The first 5 rows of the matrix A , with allowance for the notation in (4.2), (4.3), (4.7), are determined from (4.6). As the three-dimensional vectors \bar{p}_1, \bar{p}_2 in (4.6) and (4.7), we may select two unit vectors of the system of coordinates corresponding to the components of the vector \bar{r} which are the smallest in terms of modulus. The 6th row of the matrix at

$|h| > \varepsilon \mu / \tau_0$ (motion is different from parabolic motion) is determined from (4.20), and at $|h| \leq \varepsilon \mu / \tau_0$ (parabolic motion and motion close to it) -- from (4.21). Here h is the energy integral; $\varepsilon > 0$ -- a given small quantity. The quantities in (4.21) are determined from (4.22) - (4.30).

The matrices of the isochronous derivatives of second order (1.4) for the known matrices A, Φ are found from (3.3), where $a_{k\ell}$ are elements of the matrix

A^{-1} . For $\ell = 1, \dots, 5$, the matrices D^ℓ in (3.3), with allowance for the notation in (4.3), (5.1), are determined from (5.2), and $E^\ell \equiv 0$. At

$|h| > \varepsilon \mu / \tau_0$, where $\varepsilon > 0$, the same holds for the first derivatives, and the matrix D^6 is determined from (5.4), $E^6 \equiv 0$, and at $|h| \leq \varepsilon \mu / \tau_0$

the matrices D^6, E^6 -- with allowance for (4.2), (4.4), (4.22) - (4.24), (5.18) -- are determined from (5.19) - (5.22). The integral in (4.24), determined from (4.25) - (4.30), can be assumed to be already calculated when the first derivatives are found.

The matrices D_0^l, E_0^l , $l = 1, \dots, 6$ in (3.3) represent the values of D^l, E^l , at

$$\bar{\tau} = \bar{\tau}_0, \bar{U} = \bar{U}_0, t = t_0. \quad (7.1)$$

Thus, for the case $|h| \leq \varepsilon \mu / \tau_0$ (parabolic motion and motion close to it), the sum $D_0^6 + E_0^6$ in (3.3) equals the matrix (5.20) under the condition (7.1), /21 since the sum of the matrices (5.21) and (5.22) equals zero under this condition.

The first and second isochronous derivatives of $\bar{\tau}, \bar{U}$ with respect to the gravitational parameter μ for known matrices Φ, Ψ^k , $k = 1, \dots, 6$ are determined from (6.1) - (6.3).

To calculate the integrals

$$R = \int_{t_0}^t \tau dt, \quad (A.1)$$

$$\bar{R} = \int_{t_0}^t \bar{\tau} dt \quad (A.2)$$

we change to a new variable -- the control time τ , determined from the differential equation [4]

$$\frac{d\tau}{dt} = \frac{\sqrt{\mu}}{\tau}, \quad \tau(t_0) = 0. \quad (A.3)$$

We may readily obtain formulas (4.29), (4.30) from (4.22) and (A.3) for the first and second derivatives of τ with respect to τ . We find the following from (4.30)

$$\left. \begin{aligned} \tau^{(2n-1)} &= (-\alpha)^{n-1} \tau' \\ \tau^{(2n)} &= (-\alpha)^{n-1} \tau'' \end{aligned} \right\} n \geq 1, \quad (A.4)$$

where $\tau^{(k)} = \frac{d^k \tau}{d\tau^k}$, $k \geq 1$; α -- is determined by the formula (4.27). Substituting $\tau = \sqrt{\mu} \frac{dt}{d\tau}$ from (A.3) in (4.30) and taking the integrals from both sides of the equation obtained, we find (4.28) for τ .

From (A.1) and (A.3), we have

$$R(\tau) = \frac{1}{\sqrt{\mu}} \int_0^\tau \tau^2 d\tau. \quad (A.5)$$

Expanding the value $R(0) = 0$ of the function $R(\tau)$ in Taylor series at the point τ , we obtain

$$R(\tau) = -\frac{1}{\sqrt{\mu}} \sum_{n=1}^{\infty} \frac{(\tau^2)^{(n-1)}}{n!} (-\tau)^n =$$

$$= \frac{1}{\sqrt{\mu}} \left(\tau^2 \tau - \tau \tau' \tau^2 + \frac{\tau \tau'' + \tau'^2}{3} \tau^3 + \bar{\sigma} \right), \quad (\text{A.6})$$

where

$$\bar{\sigma} = \sum_{n=2}^{\infty} \left[\frac{(\tau^2)^{(2n)}}{2n+1} \tau - (\tau^2)^{(2n-1)} \right] \frac{\tau^{2n}}{(2n)!}. \quad (\text{A.7})$$

We find the derivatives in (A.7) from (4.30) and (A.4)

$$(\tau^2)^{(2n-1)} = \sum_{k=0}^{2n-1} C_{2n-1}^k \tau^{(k)} \tau^{(2n-1-k)} =$$

$$= 2(-\alpha)^{n-1} \tau \tau' + (-\alpha)^{n-2} \tau' \tau'' \sum_{k=1}^{2n-2} C_{2n-1}^k =$$

$$= 2(-\alpha)^{n-2} \tau' (2^{2(n-1)} \tau'' - 1), \quad (\text{A.8})$$

$$(\tau^2)^{(2n)} = [(\tau^2)^{(2n-1)}]' = 2(-\alpha)^{n-2} [2^{2(n-1)} (\tau''^2 - \alpha \tau'^2) - \tau'']. \quad (\text{A.9})$$

Substituting (A.8) and (A.9) in the series (A.7), we obtain the formula (4.26) for $\bar{\sigma}$. We may show that the series (4.26) converges absolutely for all τ .

For the case when $\alpha \neq 0$ (the motion differs from parabolic motion) we calculate the integral (A.1) in final form. We should note that when the equations (4.23) and (4.25) are used for the focal parameter p the following relationship holds:

$$p = \frac{C^2}{\mu} = \frac{\tau^2 v^2 - (\bar{\tau} \bar{v})^2}{\mu} = 2\tau^0 - \alpha \tau^2 - \tau'^2. \quad (\text{A.10})$$

From (4.24)

$$\int_0^{\tau} \tau'^2 d\tau = \tau \tau' - \tau_0 \tau'_0 + \alpha \int_0^{\tau} \tau^2 d\tau - \int_0^{\tau} \tau d\tau, \quad (\text{A.11})$$

and from (A.3)

$$\int_0^{\tau} \tau d\tau = \sqrt{\mu} (t - t_0) . \quad (A.12)$$

Taking the integral from both sides of the equation (A.10) considering A.11) and (A.12), according to (A.5), we obtain the final expression (4.31) for the integral /24
R.

We now find the integral (A.2). From (A.2) and (A.3), we have

$$\bar{R} = \frac{1}{\sqrt{\mu}} \int_0^{\tau} \tau \bar{\tau} d\tau . \quad (A.13)$$

From $\dot{\bar{\tau}} = \bar{v}$ and (A.3), we obtain

$$\bar{\tau}' = \frac{d\bar{\tau}}{d\tau} = \frac{\tau \bar{v}}{\sqrt{\mu}} . \quad (A.14)$$

Using (4.4), (4.3), (4.29), (A.14), we may obtain the formula

$$\frac{\bar{\Psi}}{\sqrt{\mu}} \tau = \bar{\tau} - \tau' \bar{\tau}' - \alpha \tau \bar{\tau} . \quad (A.15)$$

Expressing $\bar{\tau}$ from (4.4), according to (A.3), (A.12), we have

$$\begin{aligned} \int_0^{\tau} \bar{\tau} d\tau &= \frac{1}{\sqrt{\mu}} \int_0^{\tau} \tau \bar{v} d\tau \times \bar{C} - \frac{\bar{\Psi}}{\sqrt{\mu}} \int_0^{\tau} \tau d\tau = \\ &= \frac{1}{\sqrt{\mu}} (\bar{\tau} - \bar{\tau}_0) \times \bar{C} - \frac{\bar{\Psi}}{\sqrt{\mu}} (t - t_0) . \end{aligned} \quad (A.16)$$

Using the dependence (4.20), we calculate the integral

$$\begin{aligned} \int_0^{\tau} \tau' \bar{\tau}' d\tau &= \bar{\tau} \tau' \Big|_0^{\tau} - \int_0^{\tau} \bar{\tau} \tau'' d\tau = \\ &= \bar{\tau} \tau' - \bar{\tau}_0 \tau'_0 + \alpha \int_0^{\tau} \tau \bar{\tau} d\tau - \int_0^{\tau} \bar{\tau} d\tau . \end{aligned} \quad (A.17)$$

Taking the integral of both sides of the equation (A.15), considering (A.12), (A.13), (A.16), (A.17), we obtain:

$$2\alpha \sqrt{\mu} \bar{R} = -3 \frac{\bar{\Psi}}{\sqrt{\mu}} (t - t_0) + \frac{2}{\sqrt{\mu}} (\bar{\tau} - \bar{\tau}_0) \times \bar{C} - \bar{\tau} \tau' + \bar{\tau}_0 \tau'_0 . \quad (A.18)$$

Substituting (4.3) into (A.18) (we must substitute $\bar{C} = \bar{\gamma}_o \times \bar{U}_o$),
in the product $\bar{\gamma}_o \times \bar{C}$), (4.27) and (4.29), we reduce (A.18) to the form (5.6),
(5.7).

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